On the quantum symplectic sphere

Quantum Groups Seminar

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Sophie Emma Mikkelsen

SDU
Department of Mathematics and Computer Science

Quantum spheres

The commutative C^*-algebra $C(S^{2n+1})$ can equally be described as the universal C^*-algebra generated by $\{z_i\}_{i=0}^n$ such that $z_i z_j = z_j z_i \; \forall i, j$ and

$$\sum_{i=0}^{n} z_i z_i^* = 1.$$

QUANTUM SPHERES

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$$\sum_{i=0}^{n} z_iz_i^* = 1.$$

Let $q \in (0,1)$, we now change the relations as follows:

$$z_jz_i = qz_iz_j, \quad \text{for } i<j, \;\; z_iz_j^* = qz_j^*z_i, \quad \text{for } i \neq j,$$

$$z_i^*z_i = z_iz_i^* + (1-q^2)\sum_{j>i}^{n} z_jz_j^*, \; i = 1,\dots,n$$

$$\sum_{j=0}^{n} z_jz_j^* = 1.$$

Definition (The quantum sphere by Vaksman and Soibelman)

Let $C(S_q^{2n+1})$ be the universal C^*-algebra generated by $\{z_i\}_{i=0}^n$ and their adjoints subject to the previous relations.

S_q^{2n+1} is a non-existing *virtual* space.

We realise $C(S_q^{2n+1})$ as the algebra of *continuous functions on the quantum sphere* S_q^{2n+1}.

Graph C^*-algebras

$E = (E^0, E^1, r, s)$: a directed graph.

$C^*(E)$: the universal C^*-algebra generated by families of mutually orthogonal projections $\{P_v | \, v \in E^0\}$ and partial isometries $\{S_e | \, e \in E^1\}$ subject to the relations:

- $S_e^* S_f = 0, e \neq f$
- $S_e^* S_e = P_{r(e)}$
- $S_e S_e^* \leqslant P_{s(e)}$
- $P_v = \sum_{s(e)=v} S_e S_e^*$, if $\{e \in E^1 | \, s(e) = v\}$ is finite and nonempty.

Many properties of $C^*(E)$ can be obtained directly by the graph E.

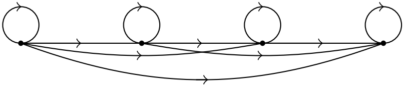
Quantum spheres as graph C^*-algebras

Let L_{2n+1} be a directed graph with $n+1$ vertices denoted $\{v_0, v_1, v_2, ..., v_n\}$ and edges $\bigcup_{i=0}^{n}\{e_{ij} \mid j = i, ..., n\}$ such that

$$s(e_{ij}) = v_i, \;\; r(e_{ij}) = v_j.$$

Due to Hong and Szymański $C(S_q^{2n+1}) \cong C^*(L_{2n+1})$.

L_7



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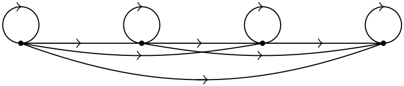

The graph C^*-algebras are independent of the parameter q.

Another quantum sphere

Another quantum sphere is constructed as a subalgebra of the *quantum symplectic group* $SP_q(n)$. (Faddeev-Takhtadzhyan-Reshetikhin, 1990)

ANOTHER QUANTUM SPHERE

Another quantum sphere is constructed as a subalgebra of the *quantum symplectic group* $SP_q(n)$. (Faddeev-Takhtadzhyan-Reshetikhin, 1990)

It is defined by the generators $\{x_i, y_i\}_{i=1}^n$ and the relations:

$$\sum_{i=1}^{n} (x_i^* x_i + y_i^* y_i) = 1,$$

$$x_i x_j = q^{-1} x_j x_i \ \forall i < j, \ \ y_i y_j = q y_j y_i \ \forall i < j, \ \ y_j x_i = q x_i y_j \ \forall i \neq j,$$
$$x_i y_i^* = q^2 y_i^* x_i, \ \ x_i x_j^* = q x_j^* x_i \ \forall i \neq j, \ \ x_i y_j^* = q y_j^* x_i \ \forall i < j,$$

$$\vdots$$

$$y_i y_j^* = q y_j^* y_i - (q^2-1) q^{2n+2-i-j} x_i^* x_j \ \forall i \neq j,$$

$$x_i y_j^* = q y_j^* x_i + (q^2-1) q^{i-j} y_i^* x_j \ \forall i > j,$$

$$y_i x_i = q^2 x_i y_i + (q^2-1) \sum_{k=1}^{i-1} q^{i-k} x_k y_k, \ \ x_i x_i^* = x_i^* x_i + (1-q^2) \sum_{k=1}^{i-1} x_k^* x_k,$$

$$y_i y_i^* = y_i^* y_i + (1-q^2) \left(q^{2(n+1-i)} x_i^* x_i + \sum_{k=1}^{n} x_k^* x_k + \sum_{k=i+1}^{n} y_k^* y_k \right).$$

The qauntum symplectic $(4n-1)$- sphere

Definition

Let $C(S_q^{4n-1})$ be the universal C^*-algebra generated by the elements $\{x_i, y_i\}_{i=1}^n$ and their adjoints subject to the previous relations.

Denote by $\mathcal{O}(S_q^{4n-1})$ the polynomial $*$-subalgebra of $C(S_q^{4n-1})$.

The virtual space S_q^{4n-1} is the *quantum symplectic sphere.*

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Goal

(1) Investigate the C^*-algebra $C(S_q^{4n-1})$.

(2) Construct a vector space basis of $\mathcal{O}(S_q^{4n-1})$.

Noncommutative principal bundles

Classical compact principal bundle:

G compact Lie group

M and B compact Hausdorff spaces

$\mu : M \times G \to M$ free continous action

$\pi : M \to B$ surjective and continuous

$$G \to M \xrightarrow{\pi} B \cong M/G$$

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$A^{coC(G)} \subseteq A$ is a subalgebra

$C(G) \to A \to A^{coC(G)}$

Quantum instanton bundles

Quantum versions of the principal bundle

$$SU(2) \to S^7 \to S^4$$

are given in the purely *algebraic* framework by:

- Bonechi, Ciccoli, Dąbrowski and Tarlini
  Total space is the quantum 7-sphere by Vaksman and Soibelman.
- Landi, Pagani and Reina
  Total space is the quantum symplectic 7-sphere.

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The C^*-algebra of the quantum symplectic sphere

For any bounded $*$-representation ψ of $C(S_q^7)$ we have $\psi(x_1) = 0$ (M-Szymański, 2020).

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The result was later generalised as follows:

Theorem (D'Andrea-Landi, 2021)

Let ψ be any bounded $*$-representaion of $C(S_q^{4n-1})$ then

$$\psi(x_i) = 0 \text{ for } 1 \leqslant i < n.$$

Hence, the generators $x_i, i = 1, ...n-1$ are all zero in $C(S_q^{4n-1})$.

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Let $y_{n+1} := x_n$ then $C(S_q^{4n-1})$ is the universal C^*-algebra generated by $y_i, i = 1, ..., n+1$ subject to the relations:

$$
\begin{aligned}
&y_{n+1}y_{n+1}^* = y_{n+1}^* y_{n+1},\\
&y_i y_j = q^{-1} y_j y_i, \forall i > j,\ i, j \neq n, n+1,\\
&y_i^* y_j = q^{-1} y_j y_i^*, \forall i \neq j,\ i, j \neq n, n+1,\\
&y_{n+1} y_n = q^{-2} y_n y_{n+1},\ y_{n+1}^* y_n = q^{-2} y_n y_{n+1}^*,\\
&\sum_{i=1}^{n+1} y_i^* y_i = 1,\\
&y_i y_i^* = y_i^* y_i + (1 - q^2) \sum_{k=i+1}^{n+1} y_k^* y_k,\\
&y_n y_n^* = y_n^* y_n + (1 - q^4) y_{n+1}^* y_{n+1}.
\end{aligned}
$$

THEOREM (M, 2021)

The C-algebra of the quantum sympletic sphere $C(S_q^{4n-1}), n \geqslant 1$ is isomorphic to the graph C*-algebra $C^*(L_{2n+1})$.*

There exists two faithful $*$-representations

$$\pi : C(S_q^{4n-1}) \to \mathcal{B}(\ell^2(\mathbb{N}^n \times \mathbb{Z})), \quad \rho : C^*(L_{2n+1}) \to \mathcal{B}(\ell^2(\mathbb{N}^n \times \mathbb{Z}))$$

such that $\phi := \rho^{-1} \circ \pi$ is an isomorphism.

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The quantum symplectic sphere $C(S_q^{4n-1})$ is isomorphic to the quantum sphere $C(S_q^{2n+1})$ by Vaksman and Soibelman.

A vector space basis

What happens in the polynomial $*$-subalgebra?

A VECTOR SPACE BASIS

WHAT HAPPENS IN THE POLYNOMIAL $*$-SUBALGEBRA?

By considering the defining relations we arrive at the following conjecture:

CONJECTURE

The following elements

$$y_1^{*m_1} \cdots y_n^{*m_n} x_1^{*k_1} \cdots x_n^{*k_n} x_{n-1}^{s_{n-1}} \cdots x_1^{s_1} y_n^{t_n} \cdots y_1^{t_1},$$
$$y_1^{*m_1} \cdots y_n^{*m_n} x_1^{*k_1} \cdots x_{n-1}^{*\,k_{n-1}} x_n^{s_n} \cdots x_1^{s_1} y_n^{t_n} \cdots y_1^{t_1},$$

with $m_j, k_j, s_j, t_j \geqslant 0, y_j^0 = y_j^{*0} = x_j^0 = x_j^{*0} = 1$*, form a vector space basis for* $\mathcal{O}(S_q^{4n-1})$.

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The Diamond lemma by Bergman

A reduction system S on $\mathbb{C}\langle X\rangle$

- $X := \{x_1, .., x_n, x_1^*, ..., x_n^*, y_1, ..., y_n, y_1^*, ..., y_n^*\}$
- $\langle X\rangle$ the free unital semigroup generated by X.
- $\mathbb{C}\langle X\rangle$ the free algebra over $\mathbb{C}$ generated by X.

$\sigma := (w_\sigma, f_\sigma) \in S$ if w_σ is the left hand side of one of the relations (except the sphere relation) and f_σ is the corresponding right hand side.

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We rewrite the sphere relation as follows:

$$x_n^* x_n = 1 - \sum_{i=1}^{n-1} x_i^* x_i - \sum_{i=1}^{n} y_i^* y_i.$$

A PARTIAL ORDER ON $\langle X \rangle$

DEFINITION

For $u, v \in \langle X \rangle$, we let $u \leqslant v$ if we can write v as a sum

$$\sum_j \alpha_j a_j, \ a_j \in \langle X \rangle, \alpha_j \in \mathbb{C}$$

by using a finite number of the reductions and $u = a_j$ for some j.

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Example In $\mathcal{O}(S_q^{11})$ we have

$$y_3 x_3 = q^2 x_3 y_3 + (q^2 - 1)(q^2 x_1 y_1 + q x_2 y_2).$$

Then $x_i y_i \leqslant y_3 x_3$ for $i = 1, 2, 3$.

- It is shown that $\leqslant$ is indeed a partial order by associating some numbers to each monomial in $\langle X \rangle$.
- Moreover, it satisfies the descending chain condition.

Example Let $n = 3$. Consider the monomial

$$u = x_3 y_3 x_3 x_1^* y_1 y_1 x_3^* x_1$$

then

$$N_1(u) = 8,\ N_2(u) = 1,$$
$$N_{3,3} = 1,\ N_{3,2} = 0,\ N_{3,1}(u) = 2,$$
$$N_4(u) = 4,\ \ N_5(u) = 0,\ \ N_6(u) = 2.$$

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The following are elements in S:

$$\begin{aligned} &(y_3 x_3,\ q^2 x_3 y_3 + (q^2 - 1)(q^2 x_1 y_1 + q x_2 y_2)), \\ &(y_2 x_2,\ q^2 x_2 y_2 + (q^2 - 1) q x_1 y_1), \\ &(y_1 x_1,\ q^2 x_1 y_1). \end{aligned}$$

Ambiguities

Definition

An *(overlap) ambiguity* of S is a 5-tuple (σ, τ, a, b, c) where $\sigma, \tau \in S$ and $a, b, c \in \langle X \rangle$ with $a, b, c \neq 1$ such that $w_\sigma = ab$ and $w_\tau = bc$.

$$\begin{array}{ccc} & abc & \\ {}^{\sigma}\swarrow & & \searrow^{\tau} \\ f_\sigma c & & a f_\tau \end{array}$$

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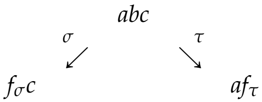
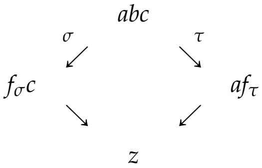


An ambiguity (σ, τ, a, b, c) is *resolvable* if $f_\sigma c$ and $a f_\tau$ can be reduced to a common expression.



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The Diamond Lemma by Bergman

Let I be the two sided ideal of $\langle X \rangle$ generated by $w_\sigma - f_\sigma$ for all $\sigma \in S$. Then

$$\mathbb{C}\langle X \rangle / I = \mathcal{O}(S_q^{4n-1}).$$

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Let $\mathbb{C}\langle X \rangle_{irr}$ be set of all elements $x \in \mathbb{C}\langle X \rangle$ such that x involves no monomials of the form

$$a w_\sigma b, \quad a, b \in \langle X \rangle, \sigma \in S.$$

Then $\mathbb{C}\langle X \rangle_{irr}$ contains precisely the elements in the conjecture.

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Then $\mathbb{C}\langle X \rangle_{irr}$ contains precisely the elements in the conjecture.

Theorem (Bergman-78)

Let $\leqslant$ be a partial order on $\langle X \rangle$ with the descending chain condition. If S is a reduction system on $\mathbb{C}\langle X \rangle$ compatible with $\leqslant$ such that all ambiguities are resolvable, then a vector space basis for $\mathbb{C}\langle X \rangle / I$ is given by $\mathbb{C}\langle X \rangle_{irr}$.

What are the ambiguities?

The ambiguities can be found by considering the sequence:

$$y_1^* y_2^* \cdots y_n^* x_1^* x_2^* \cdots x_n^* x_n \cdots x_2 x_1 y_n \cdots y_2 y_1.$$

For $n > 2$, take y_3^*, then pick an element to the left of y_3^* in the above sequence, say y_2^*. Pick then an element to the left of y_2^*, say y_1^*.

Then $y_3^* y_2^* y_1^*$ is an ambiguity.

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Then $y_3^* y_2^* y_1^*$ is an ambiguity.

We must include the following ambiguities:

$$a x_n^* x_n, \ a \in \{x_2, ..., x_1, y_n, ...y_1\},$$
$$x_n^* x_n b, \ b \in \{y_1^*, ..., y_n^*, x_1^*, ..., x_n^*\}.$$

$y_1^* y_2^* \cdots y_n^* x_1^* x_2^* \cdots x_n^* x_n \cdots x_2 x_1 y_n \cdots y_2 y_1$

ARE THE AMBIGUITIES RESOLVABLE?

When n grows larger we obtain extremely many ambiguities, making it hard to calculate by hand.

The number of ambiguities is $\frac{8}{3}(4n^3 - 3n^2 + 2n)$.

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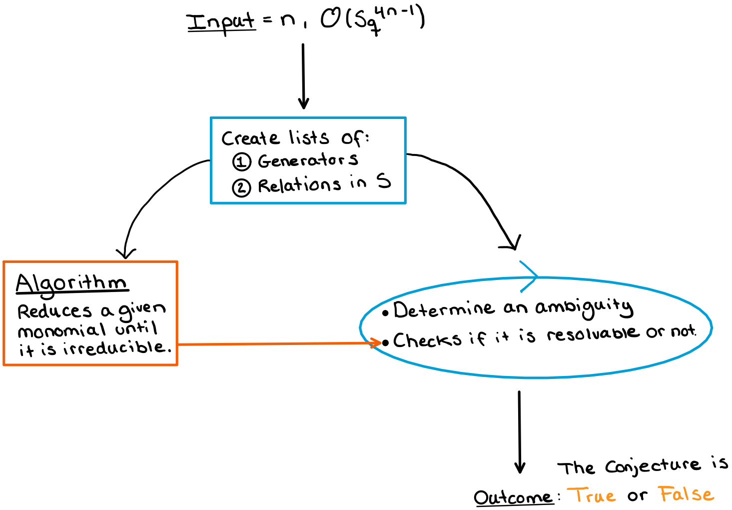
To show whether the ambiguities are resolvable or not a computer program is written in Python using the library SymPy.

$y_1^* y_2^* \cdots y_n^* x_1^* x_2^* \cdots x_n^* x_n \cdots x_2 x_1 y_n \cdots y_2 y_1$



Results

As an application of the program we conclude that the conjecture is at least true for $n = 1, 2, ..., 8$.

n	1	2	3	4	5	6	7	8
Number of ambiguities	8	64	232	576	1160	2048	3304	4992
Running time in seconds	4.80	82.29 (1.37 minutes)	509.76 (8.50 minutes)	1917.51 (32.00 minutes)	5417.61 (1.50 hour)	12944.70 (3.60 hours)	27948.76 (7.76 hours)	51751.18 (14.38 hours)

Summary of the results

(1) The generators $x_1, ..., x_{n-1}$ are zero in $C(S_q^{4n-1})$ and

$$C(S_q^{4n-1}) \cong C^*(L_{2n+1}).$$

(2) For $n = 1, .., 8$ we obtain a useful vector space basis for the algebra $\mathcal{O}(S_q^{4n-1})$.
Moreover, we conclude that $x_1, ..., x_{n-1}$ are non-zero in $\mathcal{O}(S_q^{4n-1})$.

(3) The program might become useful for other algebras or in further investigations of $\mathcal{O}(S_q^{4n-1})$.

Thank you

References

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S. E. Mikkelsen, *A vector space basis of the quantum symplectic sphere*, (2021), arXiv:2107.01406.